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Shape Preserving C^2-Spline Histopolation

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In this paper, an area matching approximation of histograms is considered under constraints like convexity, monotonicity, or positivity. Using rational-lacunary C^2-splines, sufficient conditions for the existence of convex, monotone, or positive histosplines as well as algorithms for constructing them effectively are given. Moreover, the existence criteria are shown to be satisfied for sufficiently large rationality or lacunarity parameters. © 1993 Academic Press, Inc.

1. INTRODUCTION

Let a mesh $\Delta = \{x_0, x_1, ..., x_n\}$ with $x_0 < x_1 < \cdots < x_n$ be given for the interval $[x_0, x_n]$, and let $F = \{f_1, ..., f_n\}$ be a corresponding histogram, i.e., f_i is the frequency for the interval $[x_{i-1}, x_i]$, where $i = 1, 2, ..., n$. The local mesh spacing is denoted by $h_i = x_i - x_{i-1}$. In addition let an integer $k \geqslant 1$ be chosen.

In many practical applications it is of interest to have a C^k-function s that satisfies the area matching condition

$$\int_{x_{i-1}}^{x_i} s(x)\,dx = h_i f_i, \qquad i = 1, 2, ..., n, \tag{1.1}$$

and that, in addition, preserves the shape of the given histogram F. For instance, the approximating function s should be convex, monotone, or positive if the histogram is of this kind. Here, because it is appropriate, s is assumed to be a spline function.

Shape preserving histopolation occurs in various applications. One important field is statistics. Let F be a histogram which comes from a finite sample with the observed frequency f_i in the class interval $[x_{i-1}, x_i)$, $i = 1, 2, ..., n$. Now, area matching splines can be taken as approximations to the unknown density function of the underlying random variable. Here constraints are usual. For example, an adequate approximation to the density of the exponential distribution has to be positive, monotone decreasing as well as convex. In the case of the normal distribution, besides the

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positivity the histospline is desired to be concave in the central part of the domain and convex elsewhere.

Another example of shape preserving histopolation appears in the motion of a material point. Often the problem arises: How does one choose the velocity $s = s(x)$ as a function of the time x if it is required that the point be in the positions g_i at the times x_i, $i = 0, 1, ..., n$? Because of

$$\int_{x_{i-1}}^{x_i} s(x)\, dx = g_i - g_{i-1}, \qquad i = 1, 2, ..., n,$$

in F we have to set $f_i = (g_i - g_{i-1})/h_i$, $i = 1, 2, ..., n$. In general, the positivity of the velocity is indispensable. We are led to the same model in controlling the flow of a production. Here g_i denote the output demanded at the times x_i, $i = 0, 1, ..., n$.

In data interpolation, shape preservation by splines was treated recently in several series of papers, such as [1–12], [14–16], [18–20], [22, 23], [25], and the book [27]. There are also some papers that are concerned with shape preserving histopolation; here we refer to [13, 14, 17, 20, 24] and again to the book [27]. In these references C^1-splines are used. In the present paper, shape preserving histopolation is considered by applying smoother C^2-splines.

The paper is organized as follows. First, cubic splines are chosen to discretize the histopolation problems. In convex histopolation, the result is a linear system of equalities and inequalities, while in monotone and positive histopolation these systems are nonlinear. Next, by means of optimization techniques, explicit existence criteria as well as algorithms for constructing the desired spline solutions are given. Further, if solvable at all, the histopolation problems under consideration have an infinite number of solutions. Here we propose selecting the spline solutions with minimal curvature. Finally, so-called rational–lacunary splines are treated in this context. This class extends cubic splines and, moreover, if the histogram is strictly convex, there always exist area preserving splines belonging to this class which are convex; i.e., convex histopolation is always successful in the class of rational–lacunary splines. We can derive the same property in monotone as well as in positive histopolation. Some numerical examples are given at the end of this paper.

2. Discretization of Histopolation Problems Using Cubic Splines

With the local variables $t = (x - x_{i-1})/h_i$, $u = 1 - t$, a cubic spline s can be defined for $x \in [x_{i-1}, x_i]$ by

$$s(x) = uy_{i-1} + ty_i - ut\frac{h_i^2}{6}\{(1+u)\, m_{i-1} + (1+t)m_i\}, \quad i = 1, 2, ..., n. \tag{2.1}$$

As is well known, the requirement $s'(x_i-0)=s'(x_i+0)$, $i=1, 2, ..., n-1$, leads to the C^2-condition

$$\frac{h_i}{6}m_{i-1}+\frac{h_i+h_{i+1}}{3}m_i+\frac{h_{i+1}}{6}m_{i+1}$$
$$=\frac{y_{i+1}-y_i}{h_{i+1}}-\frac{y_i-y_{i-1}}{h_i}, \qquad i=1, 2, ..., n-1, \tag{2.2}$$

and the parameters y_i and m_i have the geometrical meaning

$$y_i=s(x_i), \qquad m_i=s''(x_i), \qquad i=0, 1, ..., n. \tag{2.3}$$

For cubic splines the area matching condition reads

$$\frac{y_{i-1}+y_i}{2}-\frac{h_i^2}{24}(m_{i-1}+m_i)=f_i, \qquad i=1, 2, ..., n. \tag{2.4}$$

Further, for discretizing convexity, monotonicity, and positivity, the following proposition is important.

PROPOSITION 1. *Let s be the cubic C^2-spline given by* (2.1). *Then s is*

(i) *convex on $[x_0, x_n]$ if and only if*

$$m_i\geqslant 0, \qquad i=0, 1, ..., n; \tag{2.5}$$

(ii) *monotone increasing on $[x_0, x_n]$ if and only if*

$$\begin{gathered} n_i\geqslant 0, \quad i=0, 1, ..., n, \\ n_{i-1}-\sqrt{n_{i-1}n_i}+n_i\leqslant 3\,\frac{y_i-y_{i-1}}{h_i}, \quad i=1, 2, ..., n \end{gathered} \tag{2.6}$$

with $n_i=(y_i-y_{i-1})/h_i+h_i(m_{i-1}+2m_i)/6, i=1, 2, ..., n,$ *and with* $n_0=(y_1-y_0)/h_1-h_1(2m_0+m_1)/6$;

(iii) *nonnegative on $[x_0, x_n]$ if and only if*

$$(y_{i-1}, m_{i-1}, y_i, m_i)\in X_i\cup Y_i, \qquad i=1, 2, ..., n, \tag{2.7}$$

where $\beta_i=2y_{i-1}+y_i-h_i^2(2m_{i-1}+m_i)/6$, $\gamma_i=y_{i-1}+2y_i-h_i^2(m_{i-1}+2m_i)/6$, *and*

$$\begin{aligned} X_i&=\{(y_{i-1}, m_{i-1}, y_i, m_i): y_{i-1}\geqslant 0, y_i\geqslant 0, \beta_i\geqslant 0, \gamma_i\geqslant 0\}, \\ Y_i&=\{(y_{i-1}, m_{i-1}, y_i, m_i): 4y_{i-1}\gamma_i^3+4y_i\beta_i^3+27y_{i-1}^2y_i^2 \\ &\quad -18y_{i-1}y_i\beta_i\gamma_i-\beta_i^2\gamma_i^2\geqslant 0\}. \end{aligned} \tag{2.8}$$

The conditions (2.5), (2.6), and (2.7) are straightforward consequences of the following criteria for the nonnegativity of polynomials on fixed intervals which are derived in Ref. [23].

PROPOSITION 2. *We have*

(i) $a+bx+cx^2 \geqslant 0$ *for* $x \in [0, 1]$ *if and only if*

$$\alpha \geqslant 0, \qquad \gamma \geqslant 0, \qquad \beta \geqslant -2\sqrt{\alpha\gamma}, \tag{2.9}$$

where $\alpha = a$, $\beta = 2a+b$, $\gamma = a+b+c$;

(ii) $a+bx+cx^2+dx^3 \geqslant 0$ *for* $x \in [0, 1]$ *if and only if*

$$(\alpha, \beta, \gamma, \delta) \in X \cup Y, \tag{2.10}$$

where $\alpha = a$, $\beta = 3a+b$, $\gamma = 3a+2b+c$, $\delta = a+b+c+d$, *and*

$$X = \{(\alpha, \beta, \gamma, \delta) : \alpha \geqslant 0, \beta \geqslant 0, \gamma \geqslant 0, \delta \geqslant 0\},$$
$$Y = \{(\alpha, \beta, \gamma, \delta) : \alpha \geqslant 0, \delta \geqslant 0, 4\alpha\gamma^3 + 4\delta\beta^3 + 27\alpha^2\delta^2 - 18\alpha\beta\gamma\delta - \beta^2\gamma^2 \geqslant 0\}. \tag{2.11}$$

Note that (2.6) is equivalent to the well-known monotonicity criterion from Ref. [8].

We are now in a position to formulate the following intermediate results. The problem of convex histopolation is solvable with cubic C^2-splines if and only if there exist numbers $y_0, m_0, ..., y_n, m_n$ that satisfy the linear finite system (2.2), (2.4) under the sign condition (2.5), and, via (2.1), any solution of this system yields a convex histospline. Analogously, the solvability of the finite systems (2.2), (2.4), (2.6) and (2.2), (2.4), (2.7) ensures the existence of monotone and positive cubic C^2-spline histopolants, respectively. However, these systems are nonlinear. The cumbersome conditions (2.6) and (2.7) can be replaced by linear inequalities by sharpening them. For example,

$$0 \leqslant n_{i-1} \leqslant 3\,\frac{y_i - y_{i-1}}{h_i}, \qquad 0 \leqslant n_i \leqslant 3\,\frac{y_i - y_{i-1}}{h_i}, \qquad i = 1, 2, ..., n \tag{2.12}$$

is sufficient for (2.6), and

$$y_{i-1} \geqslant 0, \qquad y_i \geqslant 0, \qquad \beta_i \geqslant 0, \qquad \gamma_i \geqslant 0, \qquad i = 1, 2, ..., n \tag{2.13}$$

for (2.7). If the histospline is required to be convex as well as monotone increasing, then we need only add to (2.2), (2.4), (2.5) the one linear inequality

$$s'(x_0) = \frac{y_1 - y_0}{h_1} - \frac{h_1}{6}(2m_0 + m_1) \geqslant 0. \tag{2.14}$$

Finally, we mention the case in which the histospline s is required to be convex on some subintervals $[x_{i-1}, x_i]$, $i \in I_1$, and concave on other subintervals $[x_{i-1}, x_i]$, $i \in I_2$. Then we have to complete (2.2), (2.4) by

$$m_{i-1} \geqslant 0, m_i \geqslant 0 \text{ for } i \in I_1 \qquad \text{and} \qquad m_{i-1} \leqslant 0, m_i \leqslant 0 \text{ for } i \in I_2. \quad (2.15)$$

3. Existence Criteria in Convex Histopolation via Linear Programming

With $(n-1) \times (n+1)$ band matrices A, B of width 3 and $n \times (n+1)$ band matrices C, D of width 2, the system (2.2), (2.4), (2.5), which is fundamental in convex histopolation, can be written as

$$Am = By, \qquad Cm + Dy = f, \qquad m \geqslant 0. \quad (3.1)$$

The vectors of unknowns are $y = (y_0, ..., y_n)^{\mathrm{T}}$ and $m = (m_0, ..., m_n)^{\mathrm{T}}$. Without loss of generality we can assume that $f \geqslant 0$. In monotone and positive histopolation, for the condition $m \geqslant 0$ in (3.1) we have to substitute the nonlinear but convex inequalities (2.6) and (2.7), respectively.

To treat the solvability of the system (3.1), we introduce vectors of artificial variables $v = (v_1, ..., v_{n-1})^{\mathrm{T}}$ and $w = (w_1, ..., w_n)^{\mathrm{T}}$ and consider the linear program

$$\begin{aligned} &\text{minimize } z = v_1 + \cdots + v_{n-1} + w_1 + \cdots + w_n \\ &\text{subject to } Am - By + v = 0, \qquad Cm + Dy + w = f, \qquad m \geqslant 0, v \geqslant 0, w \geqslant 0. \end{aligned} \quad (3.2)$$

Since $f \geqslant 0$, there is an initial basic feasible solution, so the simplex method can be started immediately. As is well known, the program (3.2) is always solvable, and the system (3.1) is consistent whenever the optimal value $z_{\min}$ of (3.2) is equal to zero. Thus, there are convex area matching cubic C^2-splines to the given histogram F if and only if $z_{\min} = 0$, and then any optimal basic solution of program (3.2) gives a convex histospline. Analogous existence criteria can be formulated in monotone and positive histopolation, but now by means of nonlinear convex programs. In implementing this line of reasoning numerically, the question arises: How does one exploit the special structure of the program (3.2), i.e., the structure of the matrices A, B, C, and D?

3.1. *Decoupling of System* (3.1)

As a first step, in system (3.1) the vector y is eliminated to reduce the dimension. To this end, substitute the variables y_{i-1} and y_{i+1} in (2.2) by using (2.4). The result is

$$y_i = \frac{h_i f_{i+1} + h_{i+1} f_i}{h_i + h_{i+1}} - \frac{h_i h_{i+1}}{24(h_i + h_{i+1})} (h_i m_{i-1} + 3(h_i + h_{i+1}) m_i + h_{i+1} m_{i+1}),$$
$$i = 1, 2, ..., n-1. \tag{3.3}$$

Next, with y_{i-1} from (3.3), equality (2.4) yields

$$y_i = 2f_i - \frac{h_{i-1} f_i + h_i f_{i-1}}{h_{i-1} + h_i} + \frac{h_i}{24(h_{i-1} + h_i)} (h_{i-1}^2 m_{i-2} + (3h_{i-1}^2 + 5h_{i-1} h_i + 2h_i^2)\, m_{i-1} + (3h_{i-1} h_i + 2h_i^2) m_i), \qquad i = 2, ..., n. \tag{3.4}$$

Thus, (3.3) and (3.4) lead to the following system in m only,

$$a_i m_{i-2} + b_i m_{i-1} + c_i m_i + d_i m_{i+1} = e_i, \qquad i = 2, ..., n-1, \tag{3.5}$$

where we use the abbreviations

$$a_i = \frac{h_{i-1}^2}{h_{i-1} + h_i}, \quad b_i = 3h_{i-1} + \frac{2h_i + 3h_{i+1}}{h_i + h_{i+1}} h_i, \quad c_i = 3h_{i+1} + \frac{3h_{i-1} + 2h_i}{h_{i-1} + h_i} h_i,$$
$$d_i = \frac{h_{i+1}^2}{h_i + h_{i+1}}, \quad e_i = 24 \left(\frac{f_{i-1} - f_i}{h_{i-1} + h_i} - \frac{f_i - f_{i+1}}{h_i + h_{i+1}} \right). \tag{3.6}$$

This system (3.5), (2.5), which characterizes convex histopolation, can be written as

$$Em = e, \qquad m \geqslant 0, \tag{3.7}$$

where E is an $(n-2) \times (n+1)$ band matrix of width 4. As described before, the method of artificial variables yields a necessary and sufficient condition for the consistency of the system (3.7). But it is also of interest to utilize the structure of matrix E in solving the corresponding linear program.

3.2. *A Linear Program with a Bordered Diagonal Matrix*

As a second step, the system (3.7) is slightly modified. Let $\tilde{E}$ be an $(n-2) \times (n-2)$ submatrix of E built, e.g., by dropping the first, second, and last columns,

$$\tilde{E} = \begin{bmatrix} c_2 & d_2 & & & 0 \\ b_3 & \ddots & \ddots & & \\ a_4 & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & d_{n-2} \\ 0 & & a_{n-1} & b_{n-1} & c_{n-1} \end{bmatrix}. \tag{3.8}$$

For non-singular $\tilde{E}$, the system (3.7) is equivalent to

$$Gm = g, \qquad m \geqslant 0, \text{ with } G = \tilde{E}^{-1}E,\ g = \tilde{E}^{-1}e. \tag{3.9}$$

Here G is a bordered diagonal matrix,

$$G = \left[\begin{array}{c|c|ccc|c} \ & \ & \diagdown & & 0 & \ \\ & & & \diagdown & & \\ & & 0 & & \diagdown & \end{array}\right], \tag{3.10}$$

and without loss of generality we can assume that $g \geqslant 0$. Introducing the vector $w = (w_1, ..., w_{n-2})^{\mathrm{T}}$ of artificial variables we are led to the linear program

$$\begin{aligned} &\text{minimize } z = w_1 + \cdots + w_{n-2} \\ &\text{subject to } Gm + w = g, \qquad m \geqslant 0,\ w \geqslant 0, \end{aligned} \tag{3.11}$$

which yields the following existence criterion.

PROPOSITION 3. *For a given histogram F, the problem of convex histopolation is solvable with cubic C^2-splines if and only if*

$$z_{\min} = 0, \tag{3.12}$$

where $z_{\min}$ denotes the optimal value of the program (3.11). *Moreover, any optimal basic solution of* (3.11) *yields a convex histospline.*

Because of $g \geqslant 0$, the simplex method starts directly by solving (3.11). Moreover, by this method the structure of the matrix G can be completely exploited. Therefore, it is possible to solve the program (3.11) very effectively. In addition, the computation of G and g that are required to formulate (3.11) can be performed rapidly if $\tilde{E}$ is LU-factorized. Let

$$L = \begin{bmatrix} 1 & & & & 0 \\ \lambda_3 & \ddots & & & \\ \mu_4 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ 0 & & \mu_{n-1} & \lambda_{n-1} & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} v_2 & d_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & d_{n-2} \\ 0 & & & v_{n-1} \end{bmatrix}. \tag{3.13}$$

Then L and U are determined as follows:

$$\begin{aligned} &v_2 = c_2,\ \lambda_3 = b_3/v_2,\ v_3 = c_3 - \lambda_3 d_2, \text{ and for } i = 4, ..., n-1: \\ &\mu_i = a_i/v_{i-2},\ \lambda_i = (b_i - \mu_i d_{i-2})/v_{i-2},\ v_i = c_i - \lambda_i d_{i-1}. \end{aligned} \tag{3.14}$$

For $j \in \{1, 2, n+1\}$, denote by $\eta = (\eta_2, ..., \eta_{n-1})^T$ the jth column of E or e, and by $\zeta = (\zeta_2, ..., \zeta_{n-1})^T$ the jth column of G or g, respectively. Then $\zeta = \tilde{E}^{-1}\eta$ is computed by

$$t_2 = \eta_2,\ t_3 = \eta_3 - \lambda_3 t_2 \text{ and for } i = 4, ..., n-1\text{: } t_i = \eta_i - \mu_i t_{i-2} - \lambda_i t_{i-1}, \tag{3.15}$$

$$\zeta_{n-1} = t_{n-1}/v_{n-1} \text{ and for } i = n-2, ..., 2\text{: } \zeta_i = (t_i - d_i \zeta_{i+1})/v_i. \tag{3.16}$$

Thus, to formulate the essential program (3.11), it suffices to apply the formulas (3.6), (3.14), (3.15), and (3.16). Note, however, that because of the dependence on the step sizes $h_1, ..., h_n$, partial pivoting may be necessary when factorizing $\tilde{E}$. In this case, U becomes a band matrix of maximal width 4 while the width of L is unchanged.

3.3. *Stability Considerations*

It seems to be difficult to analyze the procedure described above for general step sizes. Hence we now consider the case $h_i = h$, $i = 1, 2, ..., n$. The system (3.5) then reduces to

$$\begin{aligned} &m_{i-2} + 11m_{i-1} + 11m_i + m_{i+1} \\ &\quad = \frac{24}{h^2}(f_{i-1} - 2f_i + f_{i+1}), \qquad i = 2, ..., n-1, \end{aligned} \tag{3.17}$$

i. e., $a_i = d_i = 1$ and $b_i = c_i = 11$. From (3.14) it follows that

$$v_2 = 11,\ v_3 = 10 \text{ and for } i = 4, ..., n-1\text{: } v_i = 11 - \frac{11}{v_{i-1}} + \frac{1}{v_{i-1}v_{i-2}}. \tag{3.18}$$

This recursion formula is equivalent to

$$v_2 = 11 \text{ and for } i = 3, ..., n-1\text{: } v_i = \frac{11v_{i-1} - 1}{v_{i-1} + 1}, \tag{3.19}$$

by means of which we immediately get

$$v_2 > v_3 > \cdots > v_{n-1} > v^* = 5 + \sqrt{24} \approx 9.899. \tag{3.20}$$

In view of this estimation, $\tilde{E}$ turns out to be non-singular for constant step sizes. Note that for arbitrary step sizes non-singularity of $\tilde{E}$ need not always occur.

Next, using (3.14) and (3.19), formula (3.15) is easily reformulated as

$$t_i + \left(1 + \frac{1}{v_{i-2}}\right) t_{i-1} + \frac{1}{v_{i-2}} t_{i-2} = \eta_i. \tag{3.21}$$

If v_{i-2} is approximated by v^*, the zeros of the characteristic equation corresponding to (3.21) are -1 and $-1/v^* = -0.101$. Hence, the recursion relation (3.15) is seen to be stable. Analogously, the stability of (3.16) follows.

The stability of the procedure described above depends in an essential way on the choice of the submatrix $\tilde{E}$ of E. If, in contrast to the scheme proposed above, the first three columns of E are dropped in the construction of $\tilde{E}$, then factorization becomes superfluous, i.e., $L = \tilde{E}$ and $U = I$. Instead of (3.21) we then would have

$$t_i + 11t_{i-1} + 11t_{i-2} + t_{i-3} = \eta_i. \tag{3.22}$$

But this recursion formula is unstable, at least for large n, because $-5 - \sqrt{24} \approx -9.899$ is a zero of the corresponding characteristic equation.

4. Convex Histosplines with Minimal Curvature

If the existence criterion $z_{\min} = 0$ of Proposition 3 is satisfied, then the problem of convex histopolation considered above is solvable, but in general not uniquely. Thus the question of how to select one of the spline histopolants arises. Here we propose to use the L_1-norm of a simplified geometric curvature

$$N_1(s) = \int_{x_0}^{x_n} |s''(x)| \, dx \tag{4.1}$$

as an objective function. Hence, we consider the linear program

$$\begin{aligned} &\text{minimize } N_1(s) = c^{\mathrm{T}} m \\ &\text{subject to } Gm = g, \qquad m \geqslant 0, \end{aligned} \tag{4.2}$$

where G and g are defined in (3.9) and where c is computed to be

$$c = \tfrac{1}{2}\,(h_1, h_1 + h_2, ..., h_{n-1} + h_n, h_n)^{\mathrm{T}}. \tag{4.3}$$

The linearization (4.1) of the geometric curvature can be improved if good approximations τ_i to $s'(x)$ for $x_{i-1} \leqslant x \leqslant x_i$ are known. Then, only in formula (4.3) one has to substitute $h_i/(1 + \tau_i^2)^{3/2}$ for h_i, $i = 1, 2, ..., n$.

Note that the linear programs (3.11) and (4.2) can be solved consecutively. As is well known, in the first phase of the simplex method for the program (4.2), one has to solve the program (3.11). The program (4.2) is feasible whenever $z_{\min} = 0$ in (3.11), and an optimal basic solution of (3.11) then yields an initial feasible basic solution to the program (4.2).

We remark that the L_2-norm of the curvature leads to a quadratic program while the L_∞-norm gives a linear one. But the structure of the later program is more complicated than that of the program (4.2).

5. Rational–Lacunary Splines in Shape Preserving Histopolation

The convexity test (3.12) in Proposition 3 may fail when cubic C^2-splines are used. Therefore, we are interested in extended splines that allow one to preserve convexity for suitable choices of additional parameters.

Let $p_i \geqslant 0$, $q_i \geqslant 0$ be real numbers and $k_i \geqslant 2$, $l_i \geqslant 2$ integers, $i=1, 2, ..., n$. Then, using the local variables t and u, we define for $x \in [x_{i-1}, x_i]$,

$$s(x) = uy_{i-1} + ty_i + \left(\frac{u^{k_i}}{1+p_i t} - u\right)\rho_i + \left(\frac{t^{l_i}}{1+q_i u} - t\right)\sigma_i, \qquad i=1, 2, ..., n. \tag{5.1}$$

These splines s are called rational–lacunar. Obviously, for $k_i = l_i = 3$, $p_i = q_i = 0$ the splines become cubic. In the case $k_i = l_i = 3$, $p_i \geqslant 0$, $q_i \geqslant 0$, we get the rational splines introduced in [26] while for $p_i = q_i = 0$, $k_i \geqslant 3$, $l_i \geqslant 3$ the lacunary splines from [10] are obtained. Also, the rational splines used in [9] are contained in (5.1); one sets $k_i = l_i = 2$, $p_i = q_i > 0$, and in the limit $p_i \to 0$, $i = 1, 2, ..., n$, one is led to cubic splines; see subsection 5.2.

The second derivative of the spline (5.1) is computed to be

$$s''(x) = \frac{k_i(k_i-1)(1+p_i t)^2 + 2k_i p_i u(1+p_i t) + 2p_i^2 u^2}{h_i^2(1+p_i t)^3} u^{k_i-2}\rho_i + \frac{l_i(l_i-1)(1+q_i u)^2 + 2l_i q_i t(1+q_i u) + 2q_i^2 t^2}{h_i^2(1+q_i u)^3} t^{l_i-2}\sigma_i. \tag{5.2}$$

To determine $\rho_1, \sigma_1, ..., \rho_n, \sigma_n$ such that $s''(x_i) = m_i$ holds for $i = 0, 1, ..., n$, we consider the cases $k_i \geqslant 3, l_i \geqslant 3$, $i = 1, 2, ..., n$, and $k_i = l_i = 2$, $i = 1, 2, ..., n$, separately.

5.1. *Convex Histopolation: The Case $k_i \geqslant 3$, $l_i \geqslant 3$, $p_i \geqslant 0$, $q_i \geqslant 0$*

We introduce the abbreviations

$$\varphi_i = k_i(k_i-1) + 2k_i p_i + 2p_i^2, \qquad \psi_i = l_i(l_i-1) + 2l_i q_i + 2q_i^2,$$
$$\chi_i = k_i + p_i - 1, \qquad \tau_i = l_i + q_i - 1. \tag{5.3}$$

In the case treated in this section, we obtain from (5.2)

$$\rho_i = \frac{h_i^2 m_{i-1}}{\varphi_i}, \qquad \sigma_i = \frac{h_i^2 m_i}{\psi_i}, \qquad i = 1, 2, ..., n, \tag{5.4}$$

and the C^2-condition reads

$$\frac{h_i}{\varphi_i} m_{i-1} + \left(\frac{\tau_i h_i}{\psi_i} + \frac{\chi_{i+1} h_{i+1}}{\varphi_{i+1}}\right) m_i + \frac{h_{i+1}}{\psi_{i+1}} m_{i+1}$$

$$= \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}, \qquad i = 1, 2, ..., n-1. \tag{5.5}$$

The area matching condition is reformulated as

$$\frac{y_{i-1} + y_i}{2} + \frac{2A_i - 1}{2\varphi_i} h_i^2 m_{i-1} + \frac{2B_i - 1}{2\psi_i} h_i^2 m_i = f_i, \qquad i = 1, 2, ..., n, \tag{5.6}$$

where

$$A_i = I(k_i, p_i), \qquad B_i = I(l_i, q_i),$$

$$I(k, p) = \int_0^1 \frac{(1-t)^k}{1+pt} dt = \frac{(1+p)^k}{p^{k+1}} \left\{\ln(1+p) - \sum_{\kappa=1}^{k} \frac{1}{\kappa} \left(\frac{p}{1+p}\right)^\kappa\right\}. \tag{5.7}$$

Furthermore, since $\varphi_i > 0$, $\psi_i > 0$, it is easily established by means of (5.2) and (5.4) that

$$m_i \geqslant 0, \qquad i = 0, 1, ..., n \tag{5.8}$$

is a necessary and sufficient convexity condition for the rational–lacunary splines (5.1).

Thus, in convex histopolation with rational–lacunary splines, we consider the linear system (5.5), (5.6), (5.8), which is of the type (2.2), (2.4), (2.5). Therefore, we can proceed as described before. After eliminating $y_0, ..., y_n$ in (5.5), (5.6) we get a system of the form (3.5), and it is of interest to have explicit expressions for the coefficients. These expressions are

$$\begin{aligned}
a_i &= \frac{24A_{i-1}h_{i-1}^2}{\varphi_{i-1}(h_{i-1} + h_i)}, \\
b_i &= \left(\frac{2B_{i-1} + \tau_{i-1} - 1}{\psi_{i-1}} h_{i-1} + \frac{2A_i + \chi_i - 1}{\varphi_i} h_i\right) \frac{12h_{i-1}}{h_{i-1} + h_i} \\
&\quad + \frac{(1 - 2A_i)h_i + (1 - A_i)\, h_{i+1}}{\varphi_i(h_i + h_{i+1})} 24h_i, \\
c_i &= \left(\frac{2B_i + \tau_i - 1}{\psi_i} h_i + \frac{2A_{i+1} + \chi_{i+1} - 1}{\varphi_{i+1}} h_{i+1}\right) \frac{12h_{i+1}}{h_i + h_{i+1}} \\
&\quad + \frac{(1 - B_i)\, h_{i-1} + (1 - 2B_i)\, h_i}{\psi_i(h_{i-1} + h_i)} 24h_i, \\
d_i &= \frac{24B_{i+1}h_{i+1}^2}{\psi_{i+1}(h_i + h_{i+1})}
\end{aligned} \tag{5.9}$$

while e_i is unchanged. Finally, in the present case the vector c in the program (4.2) reads

$$c=\left(\frac{\chi_1+1}{\varphi_1}h_1, \frac{\tau_1+1}{\psi_1}h_1+\frac{\chi_2+1}{\varphi_2}h_2, \ldots,\right.$$
$$\left.\frac{\tau_{n-1}+1}{\psi_{n-1}}h_{n-1}+\frac{\chi_n+1}{\varphi_n}h_n, \frac{\tau_n+1}{\psi_n}h_n\right)^{\mathrm{T}}. \tag{5.10}$$

5.2. *Convex Histopolation: The Case* $k_i=l_i=2$, $p_i\geqslant 0$, $q_i\geqslant 0$

Under these assumptions the spline (5.1) satisfies $s''(x_i)=m_i$, $i=0, 1, \ldots, n$, if

$$\rho_i=\frac{1+p_i}{2}\frac{(1+p_i)^3 m_{i-1}-m_i}{(1+p_i)^6-1}h_i^2,$$
$$\sigma_i=\frac{1+p_i}{2}\frac{(1+p_i)^3 m_i-m_{i-1}}{(1+p_i)^6-1}h_i^2, \qquad i=1, 2, \ldots, n. \tag{5.11}$$

At this point, the condition $p_i>0$, $i=1, 2, \ldots, n$, is necessary. But using, e.g., the substitution $r_i=p_i^2/(1+p_i)$, we see in a straightforward way, using

$$(1+p_i)^6-1=(1+p_i)^2(2p_i+p_i^2)(1+r_i)(3+r_i),$$
$$(1+p_it)(1+p_iu)=(1+p_i)(1+r_itu),$$
$$(1+p_i)^3(1+p_iu)-(1+p_it)=(1+p_i)(2p_i+p_i^2)(1+(1+r_i)u),$$

that the spline (5.1) can be written on $[x_{i-1}, x_i]$ as

$$s(x)=uy_{i-1}+ty_i-uth_i^2\frac{(1+(1+r_i)u)m_{i-1}+(1+(1+r_i)t)m_i}{2(1+r_i)(3+r_i)(1+r_itu)},$$
$$i=1, 2, \ldots, n. \tag{5.12}$$

Note that this representation has no singularity for $r_i=0$.

It is convenient to note that convexity with the C^2-splines (5.12) is also described by a linear system (5.5), (5.6), (5.8). Now we have to define

$$\varphi_i=\psi_i=2(1+r_i)(3+r_i), \qquad \chi_i=\tau_i=2+r_i,$$
$$A_i=B_i=J(r_i), \tag{5.13}$$
$$J(r)=\int_0^1\frac{(1-t)^3\,dt}{1+rt(1-t)}=-\frac{3}{2r}+\frac{2(3+r)}{r\sqrt{r^2+4r}}\ln\left(\frac{\sqrt{r}}{2}+\frac{\sqrt{r+4}}{2}\right),$$

and these definitions are also of interest for (5.9) and (5.10).

Thus, the procedures outlined in Sections 3 and 4 apply directly to convex histopolation with splines of the type (5.1), (5.4) as well as of the type (5.12), if the parameters p_i, q_i, k_i, l_i, r_i, $i=1, 2, ..., n$, are assumed to be fixed.

5.3. *Existence of Convex Histoplines for Large Parameters*

A histogram F on the mesh Δ is said to be in *convex position* if there exist area-true linear C^0-splines on Δ which are convex, i.e., if the system for $y_0, ..., y_n$

$$\frac{y_{i-1}+y_i}{2}=f_i, \qquad i=1, 2, ..., n, \tag{5.14}$$

$$\frac{y_i-y_{i-1}}{h_i} \leqslant \frac{y_{i+1}-y_i}{h_{i+1}}, \qquad i=1, 2, ..., n-1 \tag{5.15}$$

is solvable. For strict inequalities in (5.15), the histogram is called *strictly convex*. In this case there exists a vector $y^*=(y_0^*, ..., y_n^*)^{\mathrm{T}}$ with

$$Dy^*=f, \qquad By^*>0, \tag{5.16}$$

where B and D are band matrices,

$$B=\begin{bmatrix} \frac{1}{h_1} & -\left(\frac{1}{h_1}+\frac{1}{h_2}\right) & \frac{1}{h_2} & & \\ & \ddots & \ddots & \ddots & \\ & \frac{1}{h_{n-1}} & -\left(\frac{1}{h_{n-1}}+\frac{1}{h_n}\right) & \frac{1}{h_n} \end{bmatrix}, \tag{5.17}$$

$$D=\frac{1}{2}\begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{bmatrix}, \qquad f=\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

To reformulate (5.5), (5.6), we introduce the band matrices

$$A^*=2\begin{bmatrix} 0 & h_1+h_2 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & h_{n-1}+h_n & 0 \end{bmatrix},$$

$$R^{(r)}=\frac{1}{r+1}\begin{bmatrix} h_1 & 0 & h_2 & & \\ & \ddots & \ddots & \ddots & \\ & & h_{n-1} & 0 & h_n \end{bmatrix}, \tag{5.18}$$

$$A^{(r)}=A^*+R^{(r)},$$

$$C^{(r)}=\frac{1}{2(r+1)}\begin{bmatrix} h_1^2(2A_1-1) & h_1^2(2B_1-1) & & \\ & \ddots & \ddots & \\ & & h_n^2(2A_n-1) & h_n^2(2B_n-1) \end{bmatrix},$$

and we have

$$A^{(r)} \to A^*, \qquad C^{(r)} \to 0 \qquad \text{for} \quad r \to \infty. \tag{5.19}$$

If we assume for simplicity that

$$k_i = l_i = p_i + 3 = q_i + 3 = r + 3, \qquad i = 1, 2, ..., n \tag{5.20}$$

then the linear equations (5.5), (5.6) can be written as

$$A^{(r)}M^{(r)} = By^{(r)}, \qquad C^{(r)}M^{(r)} + Dy^{(r)} = f, \tag{5.21}$$

where $M^{(r)} = m^{(r)}/(5r+6)$.

Let $\tilde{A}^*$ be the submatrix of A^* which is obtained by dropping the first and last columns; analogously define $\tilde{A}^{(r)}$ and $\tilde{C}^{(r)}$. To obtain $\tilde{D}$, drop the first column in D; analogously define $\tilde{B}$. Now form the block matrix

$$\begin{bmatrix} \tilde{A}^* & -\tilde{B} \\ 0 & \tilde{D}, \end{bmatrix} \tag{5.22}$$

which is immediately seen to be non-singular. Thus, in view of (5.19), the matrices

$$\begin{bmatrix} \tilde{A}^{(r)} & -\tilde{B} \\ \tilde{C}^{(r)} & \tilde{D} \end{bmatrix} \tag{5.23}$$

are non-singular whenever r is sufficiently large. Therefore, system (5.21) is solvable for large r, say by $(y^{(r)}, M^{(r)})$ with $y_0^{(r)} = y_0^*$, $M_0^{(r)} = M_n^{(r)} = 0$, and we obtain $y^{(r)} \approx y^*$, $A^*M^{(r)} \approx By^*$ for these r. Because of (5.16), this implies that $M^{(r)} \geqslant 0$, and hence $m^{(r)} \geqslant 0$. Thus, we have shown that the systems (5.5), (5.6), (5.8) are solvable if, under the assumption (5.20), the real number r is sufficiently large. We remark that the same property holds true if instead of (5.20), for instance,

$$k_i = l_i = \text{constant} \geqslant 2, \qquad p_i = q_i = r, \qquad i = 1, 2, ..., n \tag{5.24}$$

or

$$p_i = q_i = \text{constant} \geqslant 0, \qquad k_i = l_i = 3 + r, \qquad i = 1, 2, ..., n \tag{5.25}$$

are assumed. We summarize these considerations as follows.

PROPOSITION 4. *Let the histogram F be strictly convex. Then there are parameters p_i, q_i, k_i, l_i, r_i, $i = 1, 2, ..., n$, such that the area preserving C^2-spline* (5.1) *with* (5.4) *or* (5.11) *is convex.*

5.4. *Existence of Positive Histosplines for Large Parameters*

In order to extend the preceding proposition to positive and monotone histopolation, we only consider the rational splines (5.12). As mentioned before, in this case the C^2 condition and the area matching condition are described by the linear systems (5.5) and (5.6) if the parameters are defined by (5.13). Next, positivity conditions are derived. To this end, substitute $t = \sigma/(1+\sigma)$ in (5.12). Then, $s(x) \geqslant 0$ for $x_{i-1} \leqslant x \leqslant x_i$ is easily seen to be equivalent to

$$y_{i-1} + \beta_i \sigma + \gamma_i \sigma^2 + y_i \sigma^3 \geqslant 0 \qquad \text{for} \quad \sigma \geqslant 0 \tag{5.26}$$

with

$$\begin{aligned} \beta_i &= (2+r_i)\, y_{i-1} + y_i - \frac{h_i^2((2+r_i)\, m_{i-1} + m_i)}{2(1+r_i)(3+r_i)}, \\ \gamma_i &= y_{i-1} + (2+r_i)\, y_i - \frac{h_i^2(m_{i-1} + (2+r_i) m_i)}{2(1+r_i)(3+r_i)}. \end{aligned} \tag{5.27}$$

Thus the linear inequalities

$$y_i \geqslant 0,\ i = 0, 1, ..., n, \qquad \beta_i \geqslant 0,\ \gamma_i \geqslant 0,\ i = 1, 2, ..., n \tag{5.28}$$

are sufficient for the positivity of the spline (5.12) on $[x_0, x_n]$.

Now, a histogram F on Δ is defined to be *in positive position* if there exist area-true linear C^0-splines on Δ which are positive, i.e., system (5.14) has solutions with

$$y_i \geqslant 0, \qquad i = 0, 1, ..., n. \tag{5.29}$$

For strict inequalities in (5.29), the histogram is said to be strictly positive. In this case, there exists a vector $y^* = (y_0^*, ..., y_n^*)^{\mathrm{T}}$ with

$$Dy^* = f, \qquad y^* > 0. \tag{5.30}$$

Here the definitions (5.17) are used.

Under the assumption

$$r_i = r, \qquad i = 1, 2, ..., n, \tag{5.31}$$

we reformulate the conditions of positive C^2-histopolation, namely (5.5), (5.6) with (5.13), and (5.28). We set $y^{(r)} = (y_0^{(r)}, ..., y_n^{(r)})^{\mathrm{T}}$ and $M^{(r)} = (M_0^{(r)}, ..., M_n^{(r)}) = m^{(r)}/(12+4r)$. Then, using the abbreviations A^*, B, D, $R^{(r)}$, and $C^{(r)}$ from (5.17) and (5.18) we get

$$A^{(r)}M^{(r)}=By^{(r)}, \qquad C^{(r)}M^{(r)}+Dy^{(r)}=f, \tag{5.32}$$

$$\begin{aligned} &y^{(r)}\geqslant 0, \qquad y_{i-1}^{(r)}+\frac{y_i^{(r)}}{2+r}-\frac{2h_i^2}{1+r}\left(M_{i-1}^{(r)}+\frac{M_i^{(r)}}{2+r}\right)\geqslant 0\\ &y_i^{(r)}+\frac{y_{i-1}^{(r)}}{2+r}-\frac{2h_i^2}{1+r}\left(M_i^{(r)}+\frac{M_{i-1}^{(r)}}{2+r}\right)\geqslant 0, \qquad i=1,2,...,n, \end{aligned} \tag{5.33}$$

where $A^{(r)}=2(2+1r)/(1+1r)A^*+2R^{(r)}$, and $C^{(r)}$ is to be replaced by $2C^{(r)}$. Since the matrices (5.23) are non-singular for sufficiently large r, the systems (5.32) are solvable, say by $(y^{(r)}, M^{(r)})$ with $y_0^{(r)}=y_0^*$ and $M_0^{(r)}=M_n^{(r)}=0$. Further, by standard arguments, we obtain $y^{(r)}\approx y^*$. In view of (5.30) this implies the validity of (5.33) whenever r is large enough. Thus we have proved the

PROPOSITION 5. *Let the histogram F be in strictly positive position. Then, for sufficiently large parameters $r_i=r$, $i=1,2,...,n$, the area matching C^2-splines (5.12) are positive on $[x_0, x_n]$.*

5.5. *Existence of Monotone Histoplines for Large Parameters*

Paralleling the preceding developments, we now discuss the monotone histopolation with C^2-splines of the type (5.12). First, we mention that $s'(x)\geqslant 0$ for $x_{i-1}\leqslant x\leqslant x_i$ is equivalent to

$$\alpha_i+2\beta_i\sigma+\gamma_i\sigma^2+2\delta_i\sigma^3+\varepsilon_i\sigma^4\geqslant 0 \qquad \text{for} \quad \sigma\geqslant 0 \tag{5.34}$$

with

$$\begin{aligned} \alpha_i&=\frac{y_i-y_{i-1}}{h_i}-\frac{h_i((2+r_i)\,m_{i-1}+m_i)}{2(1+r_i)(3+r_i)},\\ \beta_i&=(2+r_i)\frac{y_i-y_{i-1}}{h_i}-\frac{h_i(m_{i-1}+(2+r_i)m_i)}{2(1+r_i)(3+r_i)},\\ \gamma_i&=(2+r_i)^2\frac{y_i-y_{i-1}}{h_i}+\frac{h_i}{2}(m_{i-1}-m_i),\\ \delta_i&=(2+r_i)\frac{y_i-y_{i-1}}{h_i}+\frac{h_i((2+r_i)\,m_{i-1}+m_i)}{2(1+r_i)(3+r_i)},\\ \varepsilon_i&=\frac{y_i-y_{i-1}}{h_i}+\frac{h_i(m_{i-1}+(2+r_i)m_i)}{2(1+r_i)(3+r_i)}. \end{aligned} \tag{5.35}$$

Hence, a sufficient monotonicity condition reads

$$\alpha_i\geqslant 0, \qquad \beta_i\geqslant 0, \qquad \gamma_i\geqslant 0, \qquad \delta_i\geqslant 0, \qquad \varepsilon_i\geqslant 0, \qquad i=1,2,...,n, \tag{5.36}$$

and, under the assumption (5.31), the system (5.32) now has to be completed by

$$
\begin{aligned}
&y_i - y_{i-1} - \frac{2h_i^2}{1+r}((2+r)M_{i-1} + M_i) \geqslant 0, \\
&y_i - y_{i-1} - \frac{2h_i^2}{(1+r)(2+r)}(M_{i-1} + (2+r)M_i) \geqslant 0, \\
&y_i - y_{i-1} + \frac{2h_i^2(3+r)}{(2+r)^2}(M_{i-1} - M_i) \geqslant 0, \qquad (5.37)\\
&y_i - y_{i-1} + \frac{2h_i^2}{(1+r)(2+r)}((2+r)M_{i-1} + M_i) \geqslant 0, \\
&y_i - y_{i-1} + \frac{2h_i^2}{1+r}(M_{i-1} + (2+r)M_i) \geqslant 0, \qquad i = 1, 2, ..., n.
\end{aligned}
$$

A histogram F on Δ is defined to be *in strictly monotone position* if system (5.14) possesses solutions with

$$y_i - y_{i-1} > 0, \qquad i = 1, 2, ..., n. \tag{5.38}$$

Hence, there exists a vector y^* which satisfies

$$Dy^* = f, \qquad y_i^* - y_{i-1}^* \geqslant 0, \qquad i = 1, 2, ..., n. \tag{5.39}$$

In addition, there is a vector $M^* = (M_0^*, ..., M_n^*)^{\mathrm{T}}$ with the property

$$A^*M^* = By^*, \tag{5.40}$$

and M_0^*, M_n^* can be chosen according to

$$y_1^* - y_0^* - 2h_1^2 M_0^* > 0, \qquad y_n^* - y_{n-1}^* + 2h_n^2 M_n^* > 0. \tag{5.41}$$

The system (5.40) leads to

$$\frac{y_i^* - y_{i-1}^*}{h_i} + 2h_i M_i^* = \frac{y_{i+1}^* - y_i^*}{h_{i+1}} - 2h_{i+1} M_i^*, \qquad i = 1, 2, ..., n-1,$$

which, in view of (5.39), implies

$$
\begin{aligned}
&y_i^* - y_{i-1}^* + 2h_i^2 M_i^* > 0, \\
&y_{i+1}^* - y_i^* - 2h_{i+1}^2 M_i^* > 0, \qquad i = 1, 2, ..., n-1. \qquad (5.42)
\end{aligned}
$$

Now, because of the non-singularity of the matrices (5.23), the systems (5.32) are solvable for sufficiently large r, say by $(y^{(r)}, M^{(r)})$, where we can assume that $y_0^{(r)} = y_0^*$, $M_0^{(r)} = M_0^*$, and $M_n^{(r)} = M_n^*$. Continuity arguments yield $y^{(r)} \approx y^*$ and $M^{(r)} \approx M^*$. Therefore, considering (5.41) and (5.42), we see that the monotonicity condition (5.37) is satisfied for large r. Thus, we have shown the

PROPOSITION 6. *Assume that the histogram F is strictly monotone. Then the area matching C^2-splines* (5.12) *are monotone on* $[x_0, x_n]$ *if the parameters $r_i = r$, $i = 1, 2, \ldots, n$, are sufficiently large.*

6. COMPUTATIONAL COMMENTS

In a first example we consider the histogram $F = \{1, 2, M\}$ on $\Delta = \{0 < 4 < 6 < 7\}$, which is in convex position exactly when $M \geqslant 2.5$. For these M, convex area matching approximation is possible with cubic C^2-splines. For $M = 4$, Fig. 1 shows the spline which minimizes the simplified curvature (4.1). The spline shown in Fig. 2 is convex, but (4.1) is not minimal. Obviously, the first spline should be preferred.

The second histogram $F = \{M, 1, 0.5, 1, 2, M\}$ given on $\Delta = \{0 < 1 < 2 < 4 < 6 < 7 < 8\}$ is in convex position if $M \geqslant \frac{8}{3}$. In this example convex histopolation is not always successful when cubic C^2-splines are used. We found that the cubic C^2-splines are suitable for $M \geqslant 2.8$ but not for $M \leqslant 2.7$. Now, near the limit value $M = \frac{8}{3}$ rational–lacunary C^2-splines (5.1) can be taken. For fixed $k_i = l_i = 3$, $i = 1, \ldots, n$, the smallest integers $p = p_i = q_i$, $i = 1, \ldots, n$, are determined by a simple search procedure such that the convexity criterion $z_{\min} = 0$ described above is satisfied. In this way, we obtain the table

M	4	3	2.8	2.7	2.68	2.67	2.6667
p	0	0	0	1	4	14	250

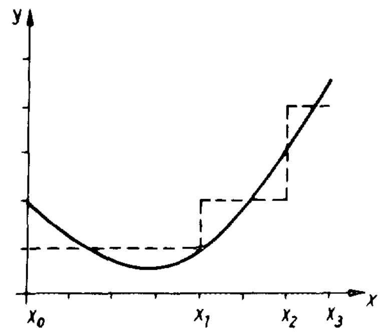

FIG. 1. Convex cubic C^2-histospline with minimized curvature (4.1).

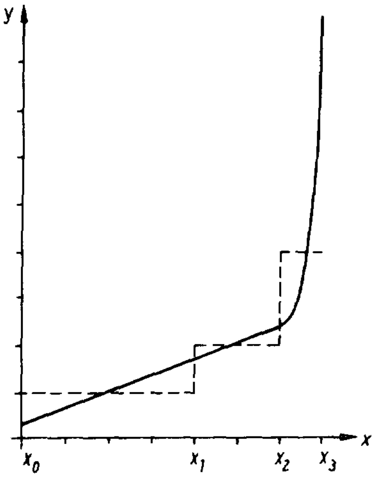

FIG. 2. Convex cubic C^2-histospline; curvature not minimized.

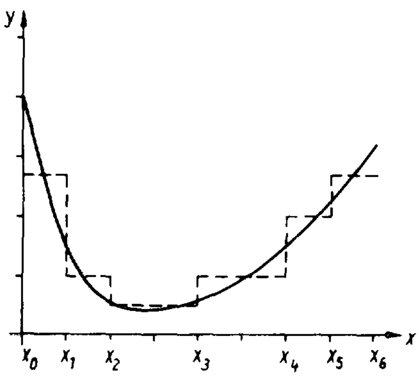

FIG. 3. Convex rational C^2-histopline for $M = 2.8$

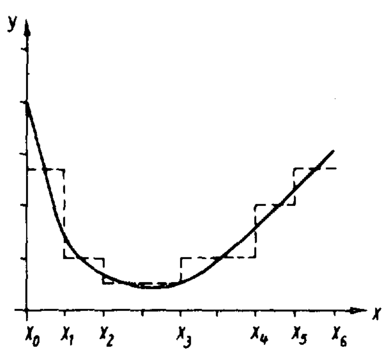

FIG. 4. Convex rational C^2-histospline for $M = 2.7$

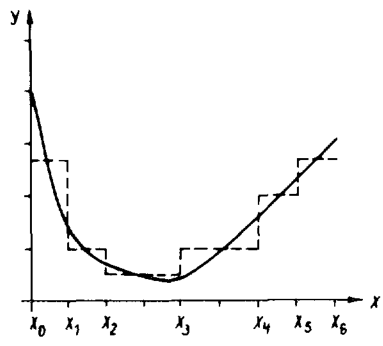

FIG. 5. Convex rational C^2-histospline for $M = 2.68$.

Figures 3–5 show the rational area matching C^2-splines which belong to the values $M = 2.8$, $M = 2.7$, and $M = 2.68$.

Finally, let us remark that also the convex histosplines (5.1) become unsatisfactory for M very close to the limit value $\frac{8}{3}$. In these cases, it is recommended that one change to a least-squares model; see, e.g., [21].

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